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On the Solution of the n-Dimensional $\oplus_B^k$ Operator

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Abstract

In this paper, we consider the solution of the equation

$$\oplus_B^k u(x) = \delta$$

where $\oplus_B^k$ is the operator related to the Bessel diamond operator iterated k-time and is defind by

$$\oplus_B^k = \left[(B_{x_1} + B_{x_2} + \cdots + B_{x_p})^4 - (B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}})^4\right]^k,$$

where $p + q = n, B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}, v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}$ [2], $x_i > 0, i = 1, 2, \ldots, n, k$ is a nonnegative integer and n is the dimension of $\mathbb{R}_n^+$. In this work we study the elementary solution of the operator $\oplus_B^k$.

Mathematics Subject Classification: 46F10

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1 Introduction

H. Yildirim, M.Z. Sarikaya and S. Ozturk [5] have first introduced the elementary solution of the n-dimensional Bessel diamond operator and the Fourier-Bessel transform of their convolution and showed that the solution of the convolution form $(-1)^k S_{2k}(x) * R_{2k}(x)$ is a unique elementary solution of the $\Diamond_B^k u(x) = \delta$.

Consider the Bessel ultra-hyperbolic operator iterated k-times,

$$\Box_B^k = \left[\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right]^k$$

Yildirim, Sarikaya and Ozturk [5] has shown that the generalized function $R_{2k}(x)$ define by (14) is the unique elementary solution of the operator $\Box_B^k$, that is $\Box_B^k R_{2k}(x) = \delta$ where $x \in \mathbb{R}_n^+ = \{x : x = (x_1, \ldots x_n), x_1 > 0, x_2 > 0, \ldots, x_n > 0\}$. Yildirim, Sarikaya and Ozturk [5] studied the Bessel diamond operator, iterated k-times,

$$\begin{aligned} \Diamond_B^k &= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right] \\ &= \left[\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right]^k \left[\sum_{i=1}^{p} B_{x_i} + \sum_{j=p+1}^{p+q} B_{x_j}\right]^k, \end{aligned} \tag{1}$$

Yildirim, Sarikaya and Ozturk [5] showed that the function $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution for the operator $\Diamond_B^k$, where $*$ indicates convolution, and $S_{2k}(x)$, $R_{2k}(x)$ are defined by (14) and (17) respectively, that is,

$$\Diamond_B^k \left((-1)^k S_{2k}(x) * R_{2k}(x)\right) = \delta(x). \tag{2}$$

Furthermore, the operator $\oplus^k$ was first studied by Kananthai, Suantai and Longani [1] . The $\oplus^k$ operator can be expressed in the form

$$\begin{aligned} \oplus^k &= \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\ &\cdot \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right]^k \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right]^k. \end{aligned} \tag{3}$$

Satsanit [4] has studied the Green function and Fourier transform for o-plus operators, iterated k-times, defined by

$$\begin{aligned}
\oplus^k &= \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k \\
&= \Diamond^k\left[\left(\frac{\triangle+\square}{2}\right)^2 + \left(\frac{\triangle-\square}{2}\right)^2\right]^k \\
&= \Diamond^k\left(\frac{\triangle^2+\square^2}{2}\right)^k \\
&= \Diamond^k \circledcirc^k .
\end{aligned} \tag{4}$$

where

$$\circledcirc^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k .$$

The purpose of this work is to study the operator

$$\begin{aligned}
\oplus_B^k &= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^4 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^4\right]^k \\
&= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k .
\end{aligned} \tag{5}$$

Let us denote the operator

$$\circledcirc_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k .$$

By (8) and (9) we obtain

$$\begin{aligned}
\circledcirc_B^k &= \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right]^k \\
&= \left[\left(\frac{\triangle_B+\square_B}{2}\right)^2 + \left(\frac{\triangle_B-\square_B}{2}\right)^2\right]^k \\
&= \left(\frac{\triangle_B^2+\square_B^2}{2}\right)^k .
\end{aligned} \tag{6}$$

Thus, (5) can be written as

$$\oplus_B^k = \diamondsuit_B^k \circledcirc_B^k = \circledcirc_B^k \diamondsuit_B^k. \tag{7}$$

For $k = 1$ the operator $\diamondsuit_B$ can be expressed in the form $\diamondsuit_B = \triangle_B \square_B = \square_B \triangle_B$ where $\square_B$ is the Bessel ultra-hyperbolic operator,

$$\square_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \cdots - B_{x_{p+q}}, \tag{8}$$

where $p + q = n$ and $\triangle_B$ is the Laplace Bessel operator,

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} + B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}}. \tag{9}$$

From (5) with $q = 0$ and $k = 1$, we obtain

$$\oplus_B = \triangle_B^4 \tag{10}$$

where

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p}. \tag{11}$$

We can find the elementary solution $u(x)$ of the operator $\oplus_B^k$; that is,

$$\oplus_B^k u(x) = \delta, \tag{12}$$

where δ is the Dirac-delta distribution. Moreover, we found that $u(x)$ relates to the elementary solution of the Laplace Bessel operator defined by (9) depending on the conditions of q and k of (5) with $q = 0$ and $k = 1$. In finding the elementary solution of (12), we use the method of convolutions of the generalized function.

2 Preliminary Notes

Denoted by T_x^y the generalized shift operator acting according to the law [2]

$$T_x^y \varphi(x) = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi\left(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta_1}, \ldots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\theta_n}\right)$$
$$\times \left(\Pi_{i=1}^n \sin^{2v_i - 1}\right) d\theta_1 \ldots d\theta_n,$$

where $x, y \in \mathbb{R}_n^+$, $C_v^* = \Pi_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [2].

$$\frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy}$$
$$U(x, 0) = f(x),$$

$$U_y(x,0)=0.$$

The convolution operator determined by T_x^y is as follow:

$$(f*\varphi)=\int_{\mathbb{R}_n^+} f(y)T_x^y\varphi(x)\left(\Pi_{i=1}^n y_i^{2v_i}\right)dy. \tag{13}$$

Convolution (13) is known as a B-convolution. We note the following properties for the B-convolution and the generalized shift operator:

(a) $T_x^y\cdot 1=1.$

(b) $T_x^0\cdot f(x)=f(x).$

(c) If $f(x),g(x)\in C(\mathbb{R}_n^+),g(x)$ is a bounded function, $x>0$ and

$$\int_0^\infty |f(x)|\left(\Pi_{i=1}^n x_i^{2v_i}\right)dx<\infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x)g(y)\left(\Pi_{i=1}^n y_i^{2v_i}\right)dy=\int_{\mathbb{R}_n^+} f(y)T_x^y g(x)\left(\Pi_{i=1}^n y_i^{2v_i}\right)dy.$$

(d) From (c), we have the following equality for $g(x)=1$,

$$\int_{\mathbb{R}_n^+} T_x^y f(x)\left(\Pi_{i=1}^n y_i^{2v_i}\right)dy=\int_{\mathbb{R}_n^+} f(y)\left(\Pi_{i=1}^n y_i^{2v_i}\right)dy$$

(e) $(f*g)(x)=(g*f)(x).$

Lemma 2.1 *Given the equation $\square_B^k u(x)=\delta(x)$ for $x\in\mathbb{R}_n^+$, where $\square_B^k$ is the Bessel-ultra hyperbolic operator iterated k-times defined by (8) . Then $u(x)=R_{2k}(x)$ is an elementary solution of the operator $\square_B^k$, where*

$$\begin{aligned}R_{2k}(x)&=\frac{V^{\frac{2k-n-|v|}{2}}}{K_n(2k)}\\&=\frac{\left(x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-x_{p+2}^2-\cdots-x_{p+q}^2\right)^{\left(\frac{2k-n-|v|}{2}\right)}}{K_n(2k)}\end{aligned} \tag{14}$$

for

$$V=x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-x_{p+2}^2-\cdots-x_{p+q}^2 \tag{15}$$

and

$$K_n(2k)=\frac{\pi^{\frac{n+2|v|-1}{2}}\Gamma\left(\frac{2+2k-n-2|v|}{2}\right)\Gamma\left(\frac{1-2k}{2}\right)\Gamma(2k)}{\Gamma\left(\frac{2+2k-p-2|v|}{2}\right)\Gamma\left(\frac{p-2k}{2}\right)}. \tag{16}$$

Lemma 2.1 *Given the equation $\triangle_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\triangle_B^k$ is the Laplace Bessel operator iterated k-times defined by (9) . Then $u(x) = (-1)^k S_{2k}(x)$ is an elementary solution of the operator $\triangle_B^k$, where*

$$S_{2k}(x) = \frac{|x|^{2k-n-2|v|}}{w_n(2k)}, \qquad p+q=n, \tag{17}$$

$$|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} \tag{18}$$

and

$$w_n(2k) = \frac{\Pi_{i=1}^n 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right)\Gamma(k)}{2^{n+2|v|-4k}\Gamma\left(\frac{n+2|v|-2k}{2}\right)}. \tag{19}$$

Lemma 2.2 *The convolution $R_{2k}(x) * (-1)^k S_{2k}(x)$ is an elementary solution for the operator $\Diamond_B^k$ iterated k-times and is defined by (1).*

Lemma 2.3 *$R_{2k}(x)$ and $S_{2k}(x)$ are homogeneous distributions of order $(2k - n - 2|v|)$.*

We need to show that $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$ satisfy the Euler equation; that is,

$$(2k-n-2|v|)\, R_{2k}(x) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_{2k}(x), \; (2k-n-2|v|)\, S_{2k}(x) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} S_{2k}(x).$$

Lemma 2.4 *(The B-convolution of tempered distribution). $R_{2k}(x) * S_{2k}(x)$ exists and is a tempered distribution.*

For the proof of Lemma 2.1- Lemma 2.4, see ([5], p.378-383).

Lemma 2.5 *The function $R_{-2k}(x)$ and $(-1)^k S_{-2k}(x)$ are the inverse in the convolution algebra of $R_{2k}(x)$ and $(-1)^k S_{2k}(x)$, respectively. That is,*

$$R_{-2k}(x) * R_{2k}(x) = R_{-2k+2k}(x) = R_0(x) = \delta(x),$$
$$(-1)^k S_{-2k}(x) * (-1)^k S_{2k}(x) = S_{-2k+2k}(x) = S_0(x) = \delta(x)$$

Lemma 2.6 *(The B-convolution of $R_{2k}(x)$ and $S_{2k}(x)$). Let $R_{2k}(x)$ and $S_{2k}(x)$ defined by (14) and (17) respectively, then we obtain:*

(1) $S_{2k}(x) * S_{2m}(x) = S_{2k+2m}(x)$, where k and m are nonnegative integers.

(2) $R_{2k}(x) * R_{2m}(x) = R_{2k+2m}(x)$, where k and m are nonnegative integers.

For the proof of Lemma 2.5 and Lemma 2.6, see [3].

3 Main Results

Theorem 3.1 *Given the equation*

$$\circledcirc_B^k G(x) = \delta(x) \tag{20}$$

for $x \in \mathbb{R}_n^+$, *where* $\circledcirc_B^k$ *is the operator iterated k-times is defined by (6). Then we obtain* $G(x)$ *is an elementary solution of (20), where*

$$G(x) = (R_{4k}(x) * (-1)^{2k} S_{4k}(x)) * (C^{*k}(x))^{*-1} \tag{21}$$

where

$$C(x) = \frac{1}{2} R_4(x) + \frac{1}{2}(-1)^2 S_4(x). \tag{22}$$

Here $C^{*k}(x)$ *denotes the convolution of* $C(x)$ *itself k-times,* $(C^{*k}(x))^{*-1}$ *denotes the inverse of* $C^{*k}(x)$ *in the convolution algebra. Moreover* $G(x)$ *is a tempered distribution.*

Proof. We have

$$\circledcirc_B^k G(x) = \left(\frac{\triangle_B^2 + \square_B^2}{2}\right)^k G(x) = \delta(x)$$

or we can write

$$\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\square_B^2\right) \left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\square_B^2\right)^{k-1} G(x) = \delta(x).$$

Convolving both sides of the above equation by $R_4(x) * (-1)^2 S_4(x)$,

$$\left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\square_B^2\right) * (R_4(x) * (-1)^2 S_4(x)) \left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\square_B^2\right)^{k-1} G(x)$$
$$= \delta(x) * R_4(x) * (-1)^2 S_4(x)$$

or

$$\left(\frac{1}{2} \triangle_B^2 (R_4(x) * (-1)^2 S_4(x)) + \frac{1}{2}\square_B^2 (R_4(x) * (-1)^2 S_4(x))\right) \left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\square_B^2\right)^{k-1} G(x)$$
$$= \delta(x) * R_4(x) * (-1)^2 S_4(x).$$

By properties of convolutions,

$$\left(\frac{1}{2} \triangle_B^2 ((-1)^2 S_4(x)) * R_4(x) + \frac{1}{2}\square_B^2 (R_4(x)) * (-1)^2 S_4(x)\right) \left(\frac{1}{2} \triangle_B^2 + \frac{1}{2}\square_B^2\right)^{k-1} G(x)$$
$$= \delta(x) * R_4(x) * (-1)^2 S_4(x).$$

By Lemma 2.1 and 2.2, we obtain

$$\left(\frac{1}{2}\delta * R_4(x) + \frac{1}{2}\delta * (-1)^2 S_4(x)\right)\left(\frac{1}{2}\triangle_B^2 + \frac{1}{2}\Box_B^2\right)^{k-1} G(x) = \delta(x) * R_4(x) * (-1)^2 S_4(x).$$

or

$$\left(\frac{1}{2}R_4(x) + \frac{1}{2}(-1)^2 S_4(x)\right)\left(\frac{1}{2}\triangle_B^2 + \frac{1}{2}\Box_B^2\right)^{k-1} G(x) = R_4(x) * (-1)^2 S_4(x).$$

Keeping on convolving both sides of the above equation by $R_4(x) * (-1)^2 S_4(x)$ up to $k-1$ times, we obtain

$$C^{*k}(x) * G(x) = \left(R_4(x) * (-1)^2 S_4(x)\right)^{*k} \tag{23}$$

the symbol $*k$ denotes the convolution of itself k-times. By properties of $R_{2k}(x)$ and $S_{2k}(x)$ in Lemma 2.7, we have

$$\left(R_4(x) * (-1)^2 S_4(x)\right)^{*k}(x) = R_{4k}(x) * (-1)^{2k} S_{4k}(x).$$

Thus (23) becomes,

$$C^{*k}(x) * G(x) = R_{4k}(x) * (-1)^{2k} S_{4k}(x)$$

or

$$G(x) = \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{24}$$

is an elementary solution of (20). We consider the function $C^{*k}(x)$, since $R_4(x) * (-1)^2 S_4(x)$ is a tempered distribution. Thus $C(x)$ defined by (22) is tempered distribution, we obtain $C^{*k}(x)$ is tempered distribution.

Now, $R_{4k}(x) * (-1)^{2k} S_{4k}(x) \in S'$, the space of tempered distribution. Choose $S' \subset D'_R$, where D'_R is the right-side distribution which is a subspace of D' of distribution. Thus $R_{4k}(x) * (-1)^{2k} S_{4k}(x) \in D'_R$. It follow that $R_{4k}(x) * (-1)^{2k} S_{4k}(x)$ is an element of convolution algebra, since D'_R is a convolution algebra. Hence Zemanian [6], (21) has a unique solution

$$G(x) = \left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1},$$

where $\left(C^{*k}(x)\right)^{*-1}$ is an inverse of $C^{*k}(x)$ in the convolution algebra. $G(x)$ is called the Green function of the operator $\circledcirc_B^k$.

Since $R_{4k}(x) * (-1)^{2k} S_{4k}(x)$ and $\left(C^{*k}(x)\right)^{*-1}$ are lies in S', then by ([6], p.152) again, we have $\left(R_{4k}(x) * (-1)^{2k} S_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \in S'$. Hence $G(x)$ is a tempered distribution. $\Box$

Theorem 3.2 *Given the equation*

$$\oplus_B^k u(x) = \delta(x), \tag{25}$$

where $\oplus_B^k$ is the operator iterated k-times defined by (5) , $\delta(x)$ is the Dirac-delta distribution, $x \in \mathbb{R}_n^+$ and k is a nonnegative integer. Then we obtain

$$u(x) = \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x) \tag{26}$$

or

$$u(x) = \left(R_{6k}(x) * (-1)^{3k} S_{6k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{27}$$

is a Green's function or an elementary solution for the operator $\oplus_B^k$ iterated k-times where $\oplus_B^k$ is defined by (5) , and $G(x)$ defined by (21) . For $q = 0$, then (25) becomes

$$\triangle_B^{4k} u(x) = \delta(x), \tag{28}$$

we obtain

$$u(x) = S_{8k}(x)$$

is an elementary solution of (28), where $\triangle_B^{4k}$ is the Laplace Bessel operator of p-dimension, iterated 4k-times and is defined by (11) . Moreover, we obtain

$$R_{-4k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right) * u(x) = R_{2k}(x) \tag{29}$$

as an elementary solution of the Bessel ultra-hyperbolic operator iterated k-times is defined by (8).

Proof. From (7) and (25), we have

$$\oplus_B^k u(x) = \left(\diamondsuit_B^k \circledcirc_B^k\right) u(x) = \delta(x). \tag{30}$$

Convolving both sides of (30) by $\left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x)$, we obtain

$$\left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x) * \left(\diamondsuit_B^k \circledcirc_B^k\right) u(x) = \delta(x) * \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x).$$

By properties of convolution

$$\diamondsuit_B^k \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * \circledcirc_B^k \left(G(x)\right) * u(x) = \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x).$$

By Lemma 2.3 and Theorem 3.1, we obtain,

$$\delta * \delta * u(x) = u(x) = \left(R_{2k}(x) * (-1)^k S_{2k}(x)\right) * G(x).$$

By Lemma 2.7 and (21), we obtain,

$$u(x) = \left(R_{6k}(x) * (-1)^{3k} S_{6k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{31}$$

is an elementary solution or Green's function of $\oplus_B^k$ operator. Now, for $q=0$ the (25) becomes

$$\triangle_B^{4k} u(x) = \delta(x), \tag{32}$$

where $\triangle_B^{4k}$ is Laplace Bessel operator of p-dimension iterated $4k$-times. By Lemma 2.2, we have

$$u(x) = (-1)^{4k} S_{8k}(x) = S_{8k}(x)$$

is an elementary solution of (28). On the other hand, we can also find $u(x)$ from (31). Since $q=0$, we have $R_{2k}(x)$ reduces to $(-1)^k S_{2k}(x)$. Thus, by (31) for $q=0$, we obtain

$$\begin{aligned} u(x) &= \left(S_{6k}(x) * (-1)^{6k} S_{6k}(x)\right) * \left((-1)^{2k} S_{4k}(x)\right)^{*-1} \\ &= (-1)^{6k} S_{6k+6k}(x) \left((-1)^{2k} S_{4k}(x)\right)^{*-1} \\ &= S_{8k}(x). \end{aligned}$$

From (31), we have

$$u(x) = \left(R_{6k}(x) * (-1)^{3k} S_{6k}(x)\right) * \left(C^{*k}(x)\right)^{*-1}.$$

Convolving the above equation by $R_{-6k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right)$. By Lemma 2.6 and 2.7, we obtain

$$R_{-6k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right) * u(x) = R_0(x) * S_0(x) * \delta(x) * R_{2k}(x)$$

or

$$R_{-6k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right) * u(x) = \delta(x) * \delta(x) * \delta(x) * R_{2k}(x).$$

It follows that

$$R_{-6k}(x) * (-1)^{3k} S_{-6k}(x) * \left(C^{*k}(x)\right) * u(x) = R_{2k}(x) \tag{33}$$

as an elementary solution of the operator $\square_B^k$ iterated k-times defined by (8) $\square$

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