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(i, j) - m_X - α -BOUNDARY AND EXTERIOR SETS IN BIMINIMAL STRUCTURE SPACES

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Abstract

The concepts of m_X - α -boundary and exterior sets in biminimal structure spaces were introduced, which found some characterizations and several properties of those sets.

1. Introduction

In general topology [7], the boundary and exterior of a subset A of a topological space X are the set of points which can be approached both from

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the closure of A and from the closure of the outside of A and the union of all open sets of X which are disjoint from A , respectively. They are the fundamental properties in topology which be more advantageous for study the concepts of topology. In 2000, Popa and Noiri [11] introduced the concepts of minimal structure spaces which included m_X -open set and m_X -closed set, specially, some characterizations and properties of those sets were found. Later, the bitopological spaces and biminimal structure spaces were introduced by Kelly [5] and then Boonpok [1], respectively. Moreover, Boonpok [1] obtained some fundamental properties of $m_X^1 m_X^2$ -closed sets and $m_X^1 m_X^2$ -open sets in biminimal structure space in 2010. Next, Boonpok [2] also introduced some notion of M -continuous functions on biminimal structure spaces and then obtained some characterizations and several properties among them. The notion of boundary and exterior sets were introduced by Sompong [12, 13], which obtained some characterizations and fundamental properties of such sets. In 2012, Carpintero et al. [4] studied preopen sets in biminimal spaces and gave some notions of among them. In 2013, Boonpok et al. [3] introduce the notion of $M_{\mathcal{A}}^{(i,j)}$ -continuous functions in biminimal structure spaces. Furthermore, they also obtain some new characterizations and several fundamental properties of $M_{\mathcal{A}}^{(i,j)}$ -continuous functions. In this study, the authors introduce the notions of (i, j) - m_X - α -boundary and (i, j) - m_X - α -exterior sets which obtain some fundamental properties of those sets in biminimal structure spaces.

2. Preliminaries

Definition 2.1 [10]. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X if $\emptyset \in m_X$ and $X \in m_X$. Each member of m_X is said to be *m_X -open* and the complement of a m_X -open set is said to be *m_X -closed*.

Definition 2.2 [6]. Let X be a nonempty set and m_X be an *m-structure*

on X . For a subset A of X , the *m_X -closure* of A and *m_X -interior* of A are defined as follows:

$$(1) m_X Cl(A) = \bigcap \{F : A \subseteq F, X \setminus F \in m_X\},$$

$$(2) m_X Int(A) = \bigcup \{U : U \subseteq A, U \in m_X\}.$$

Lemma 2.3 [6]. Let X be a nonempty set and m_X be an *m-structure* on X . For subsets A and B of X , the following hold:

$$(1) m_X Cl(X \setminus A) = X \setminus m_X Int(A) \text{ and } m_X Int(X \setminus A) = X \setminus m_X Cl(A),$$

$$(2) \text{ If } (X \setminus A) \in m_X, \text{ then } m_X Cl(A) = A \text{ and if } A \in m_X, \text{ then } m_X Int(A) = A,$$

$$(3) m_X Cl(\emptyset) = \emptyset, m_X Cl(X) = X, m_X Int(\emptyset) = \emptyset \text{ and } m_X Int(X) = X,$$

$$(4) \text{ If } A \subseteq B, \text{ then } m_X Cl(A) \subseteq m_X Cl(B) \text{ and } m_X Int(A) \subseteq m_X Int(B),$$

$$(5) A \subseteq m_X Cl(A) \text{ and } m_X Int(A) \subseteq A,$$

$$(6) m_X Cl(m_X Cl(A)) = m_X Cl(A) \text{ and } m_X Int(m_X Int(A)) = m_X Int(A).$$

Definition 2.4 [3]. Let X be a nonempty set and $m_X^1 m_X^2$ be minimal structures on X . The triple (X, m_X^1, m_X^2) is called a *bi m-space* (briefly *bispace*) or *biminimal structure space* (briefly *bimspace*).

Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . The *m_X -closure* of A and the *m_X -interior* of A with respect to m_X^i are denoted by $m_X^i Cl(A)$ and $m_X^i Int(A)$, respectively, for $i, j = 1, 2$ and $i \neq j$.

Definition 2.5 [2]. A subset A of a biminimal structure space (X, m_X^1, m_X^2) is said to be

(1) *(i, j) - m_X -regular-open* if $A = m_X^i Int(m_X^j Cl(A))$, for $i, j = 1$ or 2 and $i \neq j$,

(2) (i, j) - m_X -semi-open if $A \subseteq m_X^i Cl(m_X^j Int(A))$, for $i, j = 1$ or 2 and $i \neq j$,

(3) (i, j) - m_X -preopen if $A \subseteq m_X^i Int(m_X^j Cl(A))$, for $i, j = 1$ or 2 and $i \neq j$,

(4) (i, j) - m_X - α -open if $A \subseteq m_X^i Int(m_X^j Cl(m_X^i Int(A)))$, for $i, j = 1$ or 2 and $i \neq j$,

(5) (i, j) - m_X - β -open if $A \subseteq m_X^i Cl(m_X^j Int(m_X^i Cl(A)))$, for $i, j = 1$ or 2 and $i \neq j$.

Lemma 2.6 [2]. Let (X, m_X^1, m_X^2) be an m -space and A be a subset of X . Then

- (1) A is (i, j) - m_X -regular-closed if and only if $A \subseteq m_X^i Cl(m_X^j Int(A))$,
- (2) A is (i, j) - m_X -semi-closed if and only if $m_X^i Int(m_X^j Cl(A)) \subseteq A$,
- (3) A is (i, j) - m_X -preclosed if and only if $m_X^i Cl(m_X^j Int(A)) \subseteq A$,
- (4) A is (i, j) - m_X - α -closed if and only if $m_X^i Cl(m_X^j Int(m_X^i Cl(A))) \subseteq A$,
- (5) A is (i, j) - m_X - β -closed if and only if $m_X^i Int(m_X^j Cl(m_X^i Int(A))) \subseteq A$.

Lemma 2.7 [3]. Let (X, m_X^1, m_X^2) be a biminimal structure space and $\{A_k : k \in \mathcal{K}\}$ be a family of subsets of X .

- (1) If A_k is (i, j) - m_X - α -open for each $k \in \mathcal{K}$, then $\bigcup_{k \in \mathcal{K}} A_k$ is (i, j) - m_X - α -open.
- (2) If A_k is (i, j) - m_X - α -closed for each $k \in \mathcal{K}$, then $\bigcap_{k \in \mathcal{K}} A_k$ is (i, j) - m_X - α -closed.

Definition 2.8 [3]. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . Then, m_X^{ij} - α -closure of A and the m_X^{ij} - α -interior of A are defined as follows:

- (1) $m_X^{ij} Cl_{\mathcal{A}}(A) = \bigcap \{F : A \subseteq F, F \text{ is } (i, j)\text{-}m_X\text{-}\alpha\text{-closed}\}$,
- (2) $m_X^{ij} Int_{\mathcal{A}}(A) = \bigcup \{U : U \subseteq A, U \text{ is } (i, j)\text{-}m_X\text{-}\alpha\text{-open}\}$.

Lemma 2.9 [3]. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . The following properties hold:

- (1) $m_X^{ij} Cl_{\mathcal{A}}(A)$ is (i, j) - m_X - α -closed,
- (2) $m_X^{ij} Int_{\mathcal{A}}(A)$ is (i, j) - m_X - α -open,
- (3) A is (i, j) - m_X - α -closed if and only if $m_X^{ij} Cl_{\mathcal{A}}(A) = A$,
- (4) A is (i, j) - m_X - α -open if and only if $m_X^{ij} Int_{\mathcal{A}}(A) = A$.

Lemma 2.10 [3]. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X ,

- (1) $m_X^{ij} Cl_{\mathcal{A}}(X \setminus A) = X \setminus m_X^{ij} Int_{\mathcal{A}}(A)$,
- (2) $m_X^{ij} Int_{\mathcal{A}}(X \setminus A) = X \setminus m_X^{ij} Cl_{\mathcal{A}}(A)$.

3. (i, j) - m_X - α -boundary Sets

In this section, we define the new definitions and construct their properties of (i, j) - m_X - α -boundary sets in biminimal structure spaces.

Definition 3.1. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X and $x \in X$. Then x is called (i, j) - m_X - α -boundary point of A if $x \in m_X^{ij} Cl_{\mathcal{A}}(A) \cap m_X^{ij} Cl_{\mathcal{A}}(X \setminus A)$. We denote that the set of all (i, j) - m_X - α -boundary points of A by $m_X^{ij} Bdr_{\mathcal{A}}(A)$, where $i, j = 1, 2$ and $i \neq j$.

From Definition 3.1, $m_X^{ij}Bdr_{\mathcal{A}}(A) = m_X^{ij}Cl_{\mathcal{A}}(A) \cap m_X^{ij}Cl_{\mathcal{A}}(X \setminus A)$.

Example 3.2. Let $X = \{1, 2, 3\}$. Define m -structures m_X^1 and m_X^2 on the biminimal structure space X as follows: $m_X^1 = \{\emptyset, \{1, 2\}, X\}$ and $m_X^2 = \{\emptyset, \{2, 3\}, X\}$. By Definition 3.1, we obtain that $m_X^{ij}Bdr_{\mathcal{A}}(\{3\}) = m_X^{ij}Cl_{\mathcal{A}}(\{3\}) \cap m_X^{ij}Cl_{\mathcal{A}}(X \setminus \{3\})$. Consequently, $m_X^{12}Bdr_{\mathcal{A}}(\{3\}) = \{3\}$ and $m_X^{21}Bdr_{\mathcal{A}}(\{3\}) = X$.

Lemma 3.3. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . Then, $m_X^{ij}Bdr_{\mathcal{A}}(A) = m_X^{ij}Bdr_{\mathcal{A}}(X \setminus A)$, where $i, j = 1, 2$ and $i \neq j$.

Proof. For $i, j = 1, 2$ and $i \neq j$. By Definition 3.1, $m_X^{ij}Bdr_{\mathcal{A}}(A) = m_X^{ij}Cl_{\mathcal{A}}(A) \cap m_X^{ij}Cl_{\mathcal{A}}(X \setminus A)$ and also $m_X^{ij}Bdr_{\mathcal{A}}(X \setminus A) = m_X^{ij}Cl_{\mathcal{A}}(X \setminus A) \cap m_X^{ij}Cl_{\mathcal{A}}(X \setminus (X \setminus A)) = m_X^{ij}Cl_{\mathcal{A}}(X \setminus A) \cap m_X^{ij}Cl_{\mathcal{A}}(A)$. Consequently,

$$m_X^{ij}Bdr_{\mathcal{A}}(A) = m_X^{ij}Bdr_{\mathcal{A}}(X \setminus A),$$

where $i, j = 1, 2$ and $i \neq j$. □

Lemma 3.4. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X . Then, for any $i, j = 1, 2$ and $i \neq j$, the following statements hold:

$$(1) m_X^{ij}Bdr_{\mathcal{A}}(A) = m_X^{ij}Cl_{\mathcal{A}}(A) \setminus m_X^{ij}Int_{\mathcal{A}}(A).$$

$$(2) m_X^{ij}Bdr_{\mathcal{A}}(A) \cap m_X^{ij}Int_{\mathcal{A}}(A) = \emptyset.$$

$$(3) m_X^{ij}Bdr_{\mathcal{A}}(A) \cap m_X^{ij}Int_{\mathcal{A}}(X \setminus A) = \emptyset.$$

$$(4) m_X^{ij}Cl_{\mathcal{A}}(A) = m_X^{ij}Bdr_{\mathcal{A}}(A) \cup m_X^{ij}Int_{\mathcal{A}}(A).$$

(5) $X = m_X^{ij}Int_{\mathcal{A}}(A) \cup m_X^{ij}Bdr_{\mathcal{A}}(A) \cup m_X^{ij}Int_{\mathcal{A}}(X \setminus A)$ is a pairwise disjoint union.

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A is a subset of X .

$$\begin{aligned} (1) m_X^{ij}Bdr_{\mathcal{A}}(A) &= m_X^{ij}Cl_{\mathcal{A}}(A) \cap m_X^{ij}Cl_{\mathcal{A}}(X \setminus A) \\ &= m_X^{ij}Cl_{\mathcal{A}}(A) \cap (X \setminus m_X^{ij}Int_{\mathcal{A}}(A)) \\ &= m_X^{ij}Cl_{\mathcal{A}}(A) \setminus m_X^{ij}Int_{\mathcal{A}}(A). \end{aligned}$$

$$(2) \text{ By (1), we have that } m_X^{ij}Bdr_{\mathcal{A}}(A) \cap m_X^{ij}Int_{\mathcal{A}}(A) = \emptyset.$$

$$(3) \text{ By Lemma 3.3 and (2), we have } m_X^{ij}Bdr_{\mathcal{A}}(A) \cap m_X^{ij}Int_{\mathcal{A}}(X \setminus A) = m_X^{ij}Bdr_{\mathcal{A}}(X \setminus A) \cap m_X^{ij}Int_{\mathcal{A}}(X \setminus A) = \emptyset.$$

(4) By (1), it follows that

$$\begin{aligned} m_X^{ij}Bdr_{\mathcal{A}}(A) \cup m_X^{ij}Int_{\mathcal{A}}(A) &= (m_X^{ij}Cl_{\mathcal{A}}(A) \setminus m_X^{ij}Int_{\mathcal{A}}(A)) \cup m_X^{ij}Int_{\mathcal{A}}(A) \\ &= m_X^{ij}Cl_{\mathcal{A}}(A). \end{aligned}$$

$$\begin{aligned} (5) m_X^{ij}Int_{\mathcal{A}}(A) \cup m_X^{ij}Bdr_{\mathcal{A}}(A) \cup m_X^{ij}Int_{\mathcal{A}}(X \setminus A) \\ &= m_X^{ij}Cl_{\mathcal{A}}(A) \cup m_X^{ij}Int_{\mathcal{A}}(X \setminus A) \\ &= m_X^{ij}Cl_{\mathcal{A}}(A) \cup (X \setminus m_X^{ij}Cl_{\mathcal{A}}(A)) \\ &= X. \end{aligned}$$

By (2) and (3), we obtain that $m_X^{ij}Bdr_{\mathcal{A}}(A) \cap m_X^{ij}Int_{\mathcal{A}}(A) = \emptyset$ and $m_X^{ij}Bdr_{\mathcal{A}}(A) \cap m_X^{ij}Int_{\mathcal{A}}(X \setminus A) = \emptyset$. In order to complete the proof, we need to show that $m_X^{ij}Int_{\mathcal{A}}(A) \cap m_X^{ij}Int_{\mathcal{A}}(X \setminus A) = \emptyset$. As a result of

$m_X^{ij}Int_{\mathcal{A}} \subseteq A$ and $m_X^{ij}Int_{\mathcal{A}}(X \setminus A) \subseteq X \setminus A$. Accordingly,

$$m_X^{ij}Int_{\mathcal{A}}(A) \cap m_X^{ij}Int_{\mathcal{A}}(X \setminus A) \subseteq A \cap (X \setminus A) = \emptyset.$$

Therefore, $X = m_X^{ij}Int_{\mathcal{A}}(A) \cup m_X^{ij}Bdr_{\mathcal{A}}(A) \cup m_X^{ij}Int_{\mathcal{A}}(X \setminus A)$ is a pairwise disjoint union. The proof is completed. \square

Example 3.5. Let $X = \{1, 2, 3\}$. Define m -structures m_X^1 and m_X^2 on the biminimal structure space X as follows: $m_X^1 = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $m_X^2 = \{\emptyset, \{2\}, \{1, 3\}, X\}$.

We have that $m_X^{12}Bdr_{\mathcal{A}}(\{2, 3\}) = \emptyset$, which is a subset of $\{2, 3\}$ and $\{1\}$. But $m_X^{21}Bdr_{\mathcal{A}}(\{2, 3\}) = \{1, 3\}$ is not a subset of $\{2, 3\}$ or $\{1\}$. Thus, we need some conditions to complete the proof that $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq A$ and $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq X \setminus A$.

Theorem 3.6. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X . Then, for any $i, j = 1, 2$ and $i \neq j$, we obtain that

(1) A is (i, j) - m_X - α -closed if and only if $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq A$,

(2) A is (i, j) - m_X - α -open if and only if $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq X \setminus A$.

Proof. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X .

(1) (\Rightarrow) Assume that A is (i, j) - m_X - α -closed. That is, $m_X^{ij}Cl_{\mathcal{A}}(A) = A$.

Next, we want to show that $m_X^{ij}Bdr_{\mathcal{A}}(A) \cap (X \setminus A) = \emptyset$. By Definition 3.1, we have

$$m_X^{ij}Bdr_{\mathcal{A}}(A) \cap (X \setminus A) = (m_X^{ij}Cl_{\mathcal{A}}(A) \cap m_X^{ij}Cl_{\mathcal{A}}(X \setminus A)) \cap (X \setminus A)$$

$$= A \cap (X \setminus A)$$

$$= \emptyset.$$

Therefore, $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq A$.

(\Leftarrow) Let $m_X^{ij}Bdr_{\mathcal{A}}(A)$ be a subset of A . Then, $m_X^{ij}Bdr_{\mathcal{A}}(A) \cap (X \setminus A) = \emptyset$. Since $(X \setminus A) \subseteq m_X^{ij}Cl_{\mathcal{A}}(X \setminus A)$, $m_X^{ij}Cl_{\mathcal{A}}(A) \cap (X \setminus A) = \emptyset$, and finally, $m_X^{ij}Cl_{\mathcal{A}}(A) \subseteq A$. On the other hand, $A \subseteq m_X^{ij}Cl_{\mathcal{A}}(A)$. It follows that $m_X^{ij}Cl_{\mathcal{A}}(A) = A$. Moreover, A is (i, j) - m_X - α -closed.

(2) (\Rightarrow) Assume that A is (i, j) - m_X - α -open. Then, $m_X^{ij}Int_{\mathcal{A}}(A) = A$. Let us consider the following:

$$\begin{aligned} m_X^{ij}Bdr_{\mathcal{A}}(A) \cap A &= (m_X^{ij}Cl_{\mathcal{A}}(A) \cap m_X^{ij}Cl_{\mathcal{A}}(X \setminus A)) \cap A \\ &= A \cap m_X^{ij}Cl_{\mathcal{A}}(X \setminus A) \\ &= A \cap (X \setminus m_X^{ij}Int_{\mathcal{A}}(A)) \\ &= A \cap (X \setminus A) \\ &= \emptyset. \end{aligned}$$

Therefore, $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq X \setminus A$.

(\Leftarrow) Assume that $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq X \setminus A$. It means that $m_X^{ij}Bdr_{\mathcal{A}}(A) \cap A = \emptyset$. Since $A \subseteq m_X^{ij}Cl_{\mathcal{A}}(A)$, $X \setminus m_X^{ij}Int_{\mathcal{A}}(A) \cap A = \emptyset$, and then $A \subseteq m_X^{ij}Int_{\mathcal{A}}(A)$. On the contrary, $m_X^{ij}Int_{\mathcal{A}}(A) \subseteq A$. Consequently, $m_X^{ij}Int_{\mathcal{A}}(A) = A$. Lastly, A is (i, j) - m_X - α -open. \square

From Example 3.5, we can see that $m_X^{ij}Bdr_{\mathcal{A}}(A)$ are not necessary to be

\emptyset . For instance, $m_X^{12}Bdr_{\mathcal{A}}(\{2, 3\}) = \emptyset$, but $m_X^{21}Bdr_{\mathcal{A}}(\{2, 3\}) = \{1, 3\}$. All conditions to approach our purpose are found in the next theorem.

Theorem 3.7. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X . Then, $m_X^{ij}Bdr_{\mathcal{A}}(A) = \emptyset$ if and only if A is (i, j) - m_X - α -closed and (i, j) - m_X - α -open where $i, j = 1, 2$ and $i \neq j$.

Proof. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X .

(\Rightarrow) Assume that $m_X^{ij}Bdr_{\mathcal{A}}(A) = \emptyset$. Thus, $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq A$ and $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq X \setminus A$. By Theorem 3.6, A is (i, j) - m_X - α -closed and (i, j) - m_X - α -open.

(\Leftarrow) Assume that A is (i, j) - m_X - α -closed and (i, j) - m_X - α -open. By Theorem 3.6, we also have $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq A$ and $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq (X \setminus A)$. As a result, $m_X^{ij}Bdr_{\mathcal{A}}(A) \subseteq A \cap (X \setminus A)$, and also $m_X^{ij}Bdr_{\mathcal{A}}(A) = \emptyset$. \square

For the next section, we will introduce the concepts of (i, j) - m_X - α -exterior sets in biminimal structure space which contain some characterizations and several fundamental properties of those sets.

4. (i, j) - m_X - α -exterior Sets

Definition 4.1. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X and $x \in X$. Then, x is called (i, j) - m_X - α -exterior point of A if $x \in m_X^{ij}Int_{\mathcal{A}}(X \setminus A)$. We denote that the set of all (i, j) - m_X - α -exterior point of A by $m_X^{ij}Ext_{\mathcal{A}}(A)$, where $i, j = 1, 2$ and $i \neq j$.

According to Definition 4.1, $m_X^{ij}Ext_{\mathcal{A}}(A)$ can be rewritten as $X \setminus m_X^{ij}Cl_{\mathcal{A}}(A)$.

Example 4.2. Let $X = \{1, 2, 3\}$. Define m -structures m_X^1 and m_X^2 on the biminimal structure space X as follows: $m_X^1 = \{\emptyset, \{1, 3\}, \{2, 3\}, X\}$ and $m_X^2 = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, X\}$.

We have that $m_X^{12}Ext_{\mathcal{A}}(\{3\}) = X \setminus m_X^{12}Cl_{\mathcal{A}}(\{3\}) = X \setminus X = \emptyset$ and

$$m_X^{21}Ext_{\mathcal{A}}(\{3\}) = X \setminus m_X^{21}Cl_{\mathcal{A}}(\{3\}) = X \setminus \{3\} = \{1, 2\}.$$

Lemma 4.3. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X . Then, for any $i, j = 1, 2$ and $i \neq j$, the following statements hold:

- (1) $m_X^{ij}Ext_{\mathcal{A}}(A) \cap A = \emptyset$.
- (2) $m_X^{ij}Ext_{\mathcal{A}}(\emptyset) = X$.
- (3) $m_X^{ij}Ext_{\mathcal{A}}(X) = \emptyset$.
- (4) $m_X^{ij}Ext_{\mathcal{A}}(X \setminus m_X^{ij}Ext_{\mathcal{A}}(A)) = m_X^{ij}Ext_{\mathcal{A}}(A)$.

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A is a subset of X .

(1) For the reason that $A \subseteq m_X^{ij}Cl_{\mathcal{A}}(A)$, then $X \setminus m_X^{ij}Cl_{\mathcal{A}}(A) \subseteq X \setminus A$. Furthermore, $(X \setminus m_X^{ij}Cl_{\mathcal{A}}(A)) \cap A \subseteq \emptyset$. That is $m_X^{ij}Ext_{\mathcal{A}}(A) \cap A = \emptyset$.

(2) Since $m_X^{ij}Cl_{\mathcal{A}}(\emptyset) = \emptyset$ and by Definition 4.1, we obtain that $m_X^{ij}Ext_{\mathcal{A}}(\emptyset) = X \setminus \emptyset = X$.

(3) Similar to (2) and the fact that $m_X^{ij}Cl_{\mathcal{A}}(X) = X$, we then have $m_X^{ij}Ext_{\mathcal{A}}(X) = X \setminus X = \emptyset$.

(4) By Definition 4.1,

$$m_X^{ij} \text{Ext}_{\mathcal{A}}(X \setminus m_X^{ij} \text{Ext}_{\mathcal{A}}(A)) = m_X^{ij} \text{Ext}_{\mathcal{A}}(m_X^{ij} \text{Cl}_{\mathcal{A}}(A))$$

and

$$m_X^{ij} \text{Ext}_{\mathcal{A}}(A) = X \setminus m_X^{ij} \text{Cl}_{\mathcal{A}}(m_X^{ij} \text{Cl}_{\mathcal{A}}(A)) = m_X^{ij} \text{Ext}_{\mathcal{A}}(m_X^{ij} \text{Cl}_{\mathcal{A}}(A)).$$

Then, $m_X^{ij} \text{Ext}_{\mathcal{A}}(X \setminus m_X^{ij} \text{Ext}_{\mathcal{A}}(A)) = m_X^{ij} \text{Ext}_{\mathcal{A}}(A)$. \square

Theorem 4.4. Let (X, m_X^1, m_X^2) be a biminimal structure space and A, B be subsets of X with $A \subseteq B$. Then, $m_X^{ij} \text{Ext}_{\mathcal{A}}(B) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(A)$, where $i, j = 1, 2$ and $i \neq j$.

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A, B are subsets of X , $A \subseteq B$. Since $m_X^{ij} \text{Cl}_{\mathcal{A}}(A) \subseteq m_X^{ij} \text{Cl}_{\mathcal{A}}(B)$, we now have $X \setminus m_X^{ij} \text{Cl}_{\mathcal{A}}(B) \subseteq X \setminus m_X^{ij} \text{Cl}_{\mathcal{A}}(A)$. It follows that $m_X^{ij} \text{Ext}_{\mathcal{A}}(B) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(A)$ for any $i, j = 1, 2$ and $i \neq j$. \square

Example 4.5. Let $X = \{1, 2, 3\}$. Define m -structures m_X^1 and m_X^2 on the biminimal structure space X as follows: $m_X^1 = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $m_X^2 = \{\emptyset, \{2\}, \{1, 3\}, X\}$.

We can see that $m_X^{12} \text{Ext}_{\mathcal{A}}(\{1, 2\}) \subseteq m_X^{12} \text{Ext}_{\mathcal{A}}(\{1\})$ and $m_X^{21} \text{Ext}_{\mathcal{A}}(\{1, 2\}) \subseteq m_X^{21} \text{Ext}_{\mathcal{A}}(\{1\})$, which $\{1\}$ is a subset of $\{1, 2\}$. Moreover, we also obtain that $m_X^{12} \text{Ext}_{\mathcal{A}}(\{3\}) = \{1\} \neq X \setminus \{3\}$, $m_X^{21} \text{Ext}_{\mathcal{A}}(\{3\}) = \{2\} \neq X \setminus \{3\}$, $m_X^{12} \text{Ext}_{\mathcal{A}}(X \setminus \{3\}) = \emptyset \neq \{3\}$ and $m_X^{21} \text{Ext}_{\mathcal{A}}(X \setminus \{3\}) = \emptyset \neq \{3\}$. Therefore, we need some conditions to complete those statements.

Theorem 4.6. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . Then, for any $i, j = 1, 2$ and $i \neq j$, the following statements

are true:

(1) A is (i, j) - m_X - α -closed if and only if $m_X^{ij} \text{Ext}_{\mathcal{A}}(A) = X \setminus A$.

(2) A is (i, j) - m_X - α -open if and only if $m_X^{ij} \text{Ext}_{\mathcal{A}}(X \setminus A) = A$.

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A is a subset of X .

(1) (\Rightarrow) Suppose that A is (i, j) - m_X - α -closed. Then, $m_X^{ij} \text{Ext}_{\mathcal{A}}(A) = X \setminus m_X^{ij} \text{Cl}_{\mathcal{A}}(A) = X \setminus A$.

(\Leftarrow) Suppose that $m_X^{ij} \text{Ext}_{\mathcal{A}}(A) = X \setminus A$. It means that $X \setminus m_X^{ij} \text{Cl}_{\mathcal{A}}(A) = X \setminus A$. Since $A \subseteq m_X^{ij} \text{Cl}_{\mathcal{A}}(A)$, $m_X^{ij} \text{Cl}_{\mathcal{A}}(A) = A$. Finally, A is (i, j) - m_X - α -closed.

(2) (\Rightarrow) Suppose that A is (i, j) - m_X - α -open. Then, $X \setminus A$ is (i, j) - m_X - α -closed. Using (1), $m_X^{ij} \text{Ext}_{\mathcal{A}}(X \setminus A) = X \setminus (X \setminus A) = A$.

(\Leftarrow) Suppose that $m_X^{ij} \text{Ext}_{\mathcal{A}}(X \setminus A) = A$. We have

$$A = X \setminus m_X^{ij} \text{Cl}_{\mathcal{A}}(X \setminus A) = X \setminus (X \setminus m_X^{ij} \text{Int}_{\mathcal{A}}(A)) = m_X^{ij} \text{Int}_{\mathcal{A}}(A).$$

Hence, A is (i, j) - m_X - α -open. \square

From Example 3.5, we have $m_X^{12} \text{Ext}_{\mathcal{A}}(\{1, 2\}) \cup m_X^{12} \text{Ext}_{\mathcal{A}}(\{3\}) = \{1\} \neq m_X^{12} \text{Ext}_{\mathcal{A}}(\{1, 2\} \cap \{3\})$, whereas $m_X^{12} \text{Ext}_{\mathcal{A}}(\{1\}) \cup m_X^{12} \text{Ext}_{\mathcal{A}}(\{2, 3\}) = X = m_X^{12} \text{Ext}_{\mathcal{A}}(\{1\} \cap \{2, 3\})$. So,

$$m_X^{ij} \text{Ext}_{\mathcal{A}}(A) \cup m_X^{ij} \text{Ext}_{\mathcal{A}}(B) \text{ and } m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cap B)$$

are not necessary to be equal. In next theorem, we will give some condition for those sets to be equal. \therefore

Theorem 4.7. Let (X, m_X^1, m_X^2) be a biminimal structure space, A, B be subsets of X . Then, for any $i, j = 1, 2$ and $i \neq j$, we have

$$(1) m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cup B) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(A) \cap m_X^{ij} \text{Ext}_{\mathcal{A}}(B),$$

$$(2) m_X^{ij} \text{Ext}_{\mathcal{A}}(A) \cup m_X^{ij} \text{Ext}_{\mathcal{A}}(B) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cap B),$$

(3) if A and B are (i, j) - m_X - α -closed, then

$$m_X^{ij} \text{Ext}_{\mathcal{A}}(A) \cup m_X^{ij} \text{Ext}_{\mathcal{A}}(B) = m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cap B).$$

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space, A, B are subsets of X .

(1) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by Theorem 4.4, $m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cup B) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(A)$ and $m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cup B) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(B)$. Therefore, $m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cup B) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(A) \cap m_X^{ij} \text{Ext}_{\mathcal{A}}(B)$.

(2) By using Theorem 4.4 and the fact that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we then have $m_X^{ij} \text{Ext}_{\mathcal{A}}(A) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cap B)$ and $m_X^{ij} \text{Ext}_{\mathcal{A}}(B) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cap B)$. Lastly, $m_X^{ij} \text{Ext}_{\mathcal{A}}(A) \cup m_X^{ij} \text{Ext}_{\mathcal{A}}(B) \subseteq m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cap B)$.

(3) Assume that A and B are (i, j) - m_X - α -closed. Therefore, $A \cap B$ is also (i, j) - m_X - α -closed. By Theorem 4.6, $m_X^{ij} \text{Ext}_{\mathcal{A}}(A \cap B) = X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B) = m_X^{ij} \text{Ext}_{\mathcal{A}}(A) \cup m_X^{ij} \text{Ext}_{\mathcal{A}}(B)$. \square

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