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(i, j)-m_X-α-BOUNDARY AND EXTERIOR SETS IN BIMINIMAL STRUCTURE SPACES

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Abstract

The concepts of m_X-α-boundary and exterior sets in biminimal structure spaces were introduced, which found some characterizations and several properties of those sets.

1. Introduction

In general topology [7], the boundary and exterior of a subset A of a topological space X are the set of points which can be approached both from

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the closure of A and from the closure of the outside of A and the union of all open sets of X which are disjoint from A, respectively. They are the fundamental properties in topology which be more advantageous for study the concepts of topology. In 2000, Popa and Noiri [11] introduced the concepts of minimal structure spaces which included m_X-open set and m_X-closed set, specially, some characterizations and properties of those sets were found. Later, the bitopological spaces and biminimal structure spaces were introduced by Kelly [5] and then Boonpok [1], respectively. Moreover, Boonpok [1] obtained some fundamental properties of $m_X^1 m_X^2$-closed sets and $m_X^1 m_X^2$-open sets in biminimal structure space in 2010. Next, Boonpok [2] also introduced some notion of M-continuous functions on biminimal structure spaces and then obtained some characterizations and several properties among them. The notion of boundary and exterior sets were introduced by Sompong [12, 13], which obtained some characterizations and fundamental properties of such sets. In 2012, Carpintero et al. [4] studied preopen sets in biminimal spaces and gave some notions of among them. In 2013, Boonpok et al. [3] introduce the notion of $M_{\mathscr{A}}^{(i,j)}$-continuous functions in biminimal structure spaces. Furthermore, they also obtain some new characterizations and several fundamental properties of $M_{\mathscr{A}}^{(i,j)}$-continuous functions. In this study, the authors introduce the notions of (i, j)-m_X-α-boundary and (i, j)-m_X-α-exterior sets which obtain some fundamental properties of those sets in biminimal structure spaces.

2. Preliminaries

Definition 2.1 [10]. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X if $\varnothing \in m_X$ and $X \in m_X$. Each member of m_X is said to be m_X*-open* and the complement of a m_X-open set is said to be m_X*-closed.*

Definition 2.2 [6]. Let X be a nonempty set and m_X be an m-structure on X. For a subset A of X, the m_X*-closure* of A and m_X*-interior* of A are defined as follows:

(1) $m_X Cl(A) = \bigcap\{F : A \subseteq F,\ X \setminus F \in m_X\}$,

(2) $m_X Int(A) = \bigcup\{U : U \subseteq A,\ U \in m_X\}$.

Lemma 2.3 [6]. *Let X be a nonempty set and m_X be an m-structure on X. For subsets A and B of X, the following hold:*

(1) $m_X Cl(X\setminus A) = X\setminus m_X Int(A)$ *and* $m_X Int(X\setminus A) = X\setminus m_X Cl(A)$,

(2) *If* $(X\setminus A) \in m_X$, *then* $m_X Cl(A) = A$ *and if* $A \in m_X$, *then* $m_X Int(A) = A$,

(3) $m_X Cl(\varnothing) = \varnothing$, $m_X Cl(X) = X$, $m_X Int(\varnothing) = \varnothing$ *and* $m_X Int(X) = X$,

(4) *If* $A \subseteq B$, *then* $m_X Cl(A) \subseteq m_X Cl(B)$ *and* $m_X Int(A) \subseteq m_X Int(B)$,

(5) $A \subseteq m_X Cl(A)$ *and* $m_X Int(A) \subseteq A$,

(6) $m_X Cl(m_X Cl(A)) = m_X Cl(A)$ *and* $m_X Int(m_X Int(A)) = m_X Int(A)$.

Definition 2.4 [3]. Let X be a nonempty set and $m_X^1 m_X^2$ be minimal structures on X. The triple (X, m_X^1, m_X^2) is called a *bi m-space* (briefly *bispace*) or *biminimal structure space* (briefly *bimspace*).

Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X. The m_X-closure of A and the m_X-interior of A with respect to m_X^i are denoted by $m_X^i Cl(A)$ and $m_X^i Int(A)$, respectively, for $i, j = 1, 2$ and $i \neq j$.

Definition 2.5 [2]. A subset A of a biminimal structure space (X, m_X^1, m_X^2) is said to be

(1) (i, j)-m_X*-regular-open* if $A = m_X^i Int(m_X^j Cl(A))$, for $i, j = 1$ or 2 and $i \neq j$,

(2) (i, j)-m_X-*semi-open* if $A \subseteq m_X^i Cl(m_X^j Int(A))$, for $i, j = 1$ or 2 and $i \neq j$,

(3) (i, j)-m_X-*preopen* if $A \subseteq m_X^i Int(m_X^j Cl(A))$, for $i, j = 1$ or 2 and $i \neq j$,

(4) (i, j)-m_X-α-*open* if $A \subseteq m_X^i Int(m_X^j Cl(m_X^i Int(A)))$, for $i, j = 1$ or 2 and $i \neq j$,

(5) (i, j)-m_X-β-*open* if $A \subseteq m_X^i Cl(m_X^j Int(m_X^i Cl(A)))$, for $i, j = 1$ or 2 and $i \neq j$.

Lemma 2.6 [2]. *Let (X, m_X^1, m_X^2) be an m-space and A be a subset of X. Then*

(1) *A is (i, j)-m_X-regular-closed if and only if $A \subseteq m_X^i Cl(m_X^j Int(A))$,*

(2) *A is (i, j)-m_X-semi-closed if and only if $m_X^i Int(m_X^j Cl(A)) \subseteq A$,*

(3) *A is (i, j)-m_X-preclosed if and only if $m_X^i Cl(m_X^j Int(A)) \subseteq A$,*

(4) *A is (i, j)-m_X-α-closed if and only if $m_X^i Cl(m_X^j Int(m_X^i Cl(A))) \subseteq A$,*

(5) *A is (i, j)-m_X-β-closed if and only if $m_X^i Int(m_X^j Cl(m_X^i Int(A))) \subseteq A$.*

Lemma 2.7 [3]. *Let (X, m_X^1, m_X^2) be a biminimal structure space and $\{A_k : k \in \mathscr{K}\}$ be a family of subsets of X.*

(1) *If A_k is (i, j)-m_X-α-open for each $k \in \mathscr{K}$, then $\bigcup_{k\in\mathscr{K}} A_k$ is (i, j)-m_X-α-open.*

(2) *If A_k is (i, j)-m_X-α-closed for each $k \in \mathscr{K}$, then $\bigcap_{k\in\mathscr{K}} A_k$ is (i, j)-m_X-α-closed.*

Definition 2.8 [3]. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X. Then, m_X^{ij}-α-*closure* of A and the m_X^{ij}-α-*interior* of A are defined as follows:

(1) $m_X^{ij} Cl_{\mathscr{A}}(A) = \bigcap\{F : A \subseteq F, F \text{ is } (i, j)\text{-}m_X\text{-}\alpha\text{-}closed\}$,

(2) $m_X^{ij} Int_{\mathscr{A}}(A) = \bigcup\{U : U \subseteq A, U \text{ is } (i, j)\text{-}m_X\text{-}\alpha\text{-}open\}$.

Lemma 2.9 [3]. *Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X. The following properties hold:*

(1) *$m_X^{ij} Cl_{\mathscr{A}}(A)$ is (i, j)-m_X-α-closed,*

(2) *$m_X^{ij} Int_{\mathscr{A}}(A)$ is (i, j)-m_X-α-open,*

(3) *A is (i, j)-m_X-α-closed if and only if $m_X^{ij} Cl_{\mathscr{A}}(A) = A$,*

(4) *A is (i, j)-m_X-α-open if and only if $m_X^{ij} Int_{\mathscr{A}}(A) = A$.*

Lemma 2.10 [3]. *Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X,*

(1) $m_X^{ij} Cl_{\mathscr{A}}(X \setminus A) = X \setminus m_X^{ij} Int_{\mathscr{A}}(A)$,

(2) $m_X^{ij} Int_{\mathscr{A}}(X \setminus A) = X \setminus m_X^{ij} Cl_{\mathscr{A}}(A)$.

3. (i, j)-m_X-α-boundary Sets

In this section, we define the new definitions and construct their properties of (i, j)-m_X-α-boundary sets in biminimal structure spaces.

Definition 3.1. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X and $x \in X$. Then x is called (i, j)-m_X-α-*boundary point* of A if $x \in m_X^{ij} Cl_{\mathscr{A}}(A) \cap m_X^{ij} Cl_{\mathscr{A}}(X \setminus A)$. We denote that the set of all (i, j)-m_X-α-boundary points of A by $m_X^{ij} Bdr_{\mathscr{A}}(A)$, where $i, j = 1, 2$ and $i \neq j$.

From Definition 3.1, $m_X^{ij}Bdr_{\mathscr{A}}(A) = m_X^{ij}Cl_{\mathscr{A}}(A) \cap m_X^{ij}Cl_{\mathscr{A}}(X \setminus A)$.

Example 3.2. Let $X = \{1, 2, 3\}$. Define m-structures m_X^1 and m_X^2 on the biminimal structure space X as follows: $m_X^1 = \{\varnothing, \{1, 2\}, X\}$ and $m_X^2 = \{\varnothing, \{2, 3\}, X\}$. By Definition 3.1, we obtain that $m_X^{ij}Bdr_{\mathscr{A}}(\{3\}) = m_X^{ij}Cl_{\mathscr{A}}(\{3\}) \cap m_X^{ij}Cl_{\mathscr{A}}(X \setminus \{3\})$. Consequently, $m_X^{12}Bdr_{\mathscr{A}}(\{3\}) = \{3\}$ and $m_X^{21}Bdr_{\mathscr{A}}(\{3\}) = X$.

Lemma 3.3. *Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X. Then, $m_X^{ij}Bdr_{\mathscr{A}}(A) = m_X^{ij}Bdr_{\mathscr{A}}(X \setminus A)$, where $i, j = 1, 2$ and $i \neq j$.*

Proof. For $i, j = 1, 2$ and $i \neq j$. By Definition 3.1, $m_X^{ij}Bdr_{\mathscr{A}}(A) = m_X^{ij}Cl_{\mathscr{A}}(A) \cap m_X^{ij}Cl_{\mathscr{A}}(X \setminus A)$ and also $m_X^{ij}Bdr_{\mathscr{A}}(X \setminus A) = m_X^{ij}Cl_{\mathscr{A}}(X \setminus A) \cap m_X^{ij}Cl_{\mathscr{A}}(X \setminus (X \setminus A)) = m_X^{ij}Cl_{\mathscr{A}}(X \setminus A) \cap m_X^{ij}Cl_{\mathscr{A}}(A)$. Consequently,

$$m_X^{ij}Bdr_{\mathscr{A}}(A) = m_X^{ij}Bdr_{\mathscr{A}}(X \setminus A),$$

where $i, j = 1, 2$ and $i \neq j$. □

Lemma 3.4. *Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X. Then, for any $i, j = 1, 2$ and $i \neq j$, the following statements hold:*

(1) $m_X^{ij}Bdr_{\mathscr{A}}(A) = m_X^{ij}Cl_{\mathscr{A}}(A) \setminus m_X^{ij}Int_{\mathscr{A}}(A)$.

(2) $m_X^{ij}Bdr_{\mathscr{A}}(A) \cap m_X^{ij}Int_{\mathscr{A}}(A) = \varnothing$.

(3) $m_X^{ij}Bdr_{\mathscr{A}}(A) \cap m_X^{ij}Int_{\mathscr{A}}(X \setminus A) = \varnothing$.

(4) $m_X^{ij}Cl_{\mathscr{A}}(A) = m_X^{ij}Bdr_{\mathscr{A}}(A) \cup m_X^{ij}Int_{\mathscr{A}}(A)$.

(5) *$X = m_X^{ij}Int_{\mathscr{A}}(A) \cup m_X^{ij}Bdr_{\mathscr{A}}(A) \cup m_X^{ij}Int_{\mathscr{A}}(X \setminus A)$ is a pairwise disjoint union.*

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A is a subset of X.

(1) $m_X^{ij}Bdr_{\mathscr{A}}(A) = m_X^{ij}Cl_{\mathscr{A}}(A) \cap m_X^{ij}Cl_{\mathscr{A}}(X \setminus A)$

$= m_X^{ij}Cl_{\mathscr{A}}(A) \cap (X \setminus m_X^{ij}Int_{\mathscr{A}}(A))$

$= m_X^{ij}Cl_{\mathscr{A}}(A) \setminus m_X^{ij}Int_{\mathscr{A}}(A)$.

(2) By (1), we have that $m_X^{ij}Bdr_{\mathscr{A}}(A) \cap m_X^{ij}Int_{\mathscr{A}}(A) = \varnothing$.

(3) By Lemma 3.3 and (2), we have $m_X^{ij}Bdr_{\mathscr{A}}(A) \cap m_X^{ij}Int_{\mathscr{A}}(X \setminus A) = m_X^{ij}Bdr_{\mathscr{A}}(X \setminus A) \cap m_X^{ij}Int_{\mathscr{A}}(X \setminus A) = \varnothing$.

(4) By (1), it follows that

$$m_X^{ij}Bdr_{\mathscr{A}}(A) \cup m_X^{ij}Int_{\mathscr{A}}(A) = (m_X^{ij}Cl_{\mathscr{A}}(A) \setminus m_X^{ij}Int_{\mathscr{A}}(A)) \cup m_X^{ij}Int_{\mathscr{A}}(A)$$

$$= m_X^{ij}Cl_{\mathscr{A}}(A).$$

(5) $m_X^{ij}Int_{\mathscr{A}}(A) \cup m_X^{ij}Bdr_{\mathscr{A}}(A) \cup m_X^{ij}Int_{\mathscr{A}}(X \setminus A)$

$= m_X^{ij}Cl_{\mathscr{A}}(A) \cup m_X^{ij}Int_{\mathscr{A}}(X \setminus A)$

$= m_X^{ij}Cl_{\mathscr{A}}(A) \cup (X \setminus m_X^{ij}Cl_{\mathscr{A}}(A))$

$= X$.

By (2) and (3), we obtain that $m_X^{ij}Bdr_{\mathscr{A}}(A) \cap m_X^{ij}Int_{\mathscr{A}}(A) = \varnothing$ and $m_X^{ij}Bdr_{\mathscr{A}}(A) \cap m_X^{ij}Int_{\mathscr{A}}(X \setminus A) = \varnothing$. In order to complete the proof, we need to show that $m_X^{ij}Int_{\mathscr{A}}(A) \cap m_X^{ij}Int_{\mathscr{A}}(X \setminus A) = \varnothing$. As a result of

$m_X^{ij}Int_{\mathscr{A}} \subseteq A$ and $m_X^{ij}Int_{\mathscr{A}}(X\backslash A) \subseteq X\backslash A$. Accordingly,

$$m_X^{ij}Int_{\mathscr{A}}(A) \cap m_X^{ij}Int_{\mathscr{A}}(X\backslash A) \subseteq A \cap (X\backslash A) = \varnothing.$$

Therefore, $X = m_X^{ij}Int_{\mathscr{A}}(A) \cup m_X^{ij}Bdr_{\mathscr{A}}(A) \cup m_X^{ij}Int_{\mathscr{A}}(X\backslash A)$ is a pairwise disjoint union. The proof is completed. □

Example 3.5. Let $X = \{1, 2, 3\}$. Define m-structures m_X^1 and m_X^2 on the biminimal structure space X as follows: $m_X^1 = \{\varnothing, \{1\}, \{2, 3\}, X\}$ and $m_X^2 = \{\varnothing, \{2\}, \{1, 3\}, X\}$.

We have that $m_X^{12}Bdr_{\mathscr{A}}(\{2, 3\}) = \varnothing$, which is a subset of $\{2, 3\}$ and $\{1\}$. But $m_X^{21}Bdr_{\mathscr{A}}(\{2, 3\}) = \{1, 3\}$ is not a subset of $\{2, 3\}$ or $\{1\}$. Thus, we need some conditions to complete the proof that $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq A$ and $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq X\backslash A$.

Theorem 3.6. *Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X. Then, for any $i, j = 1, 2$ and $i \neq j$, we obtain that*

(1) A is (i, j)-m_X-α-closed if and only if $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq A$,

(2) A is (i, j)-m_X-α-open if and only if $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq X\backslash A$.

Proof. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X.

(1) ($\Rightarrow$) Assume that A is (i, j)-m_X-α-closed. That is, $m_X^{ij}Cl_{\mathscr{A}}(A) = A$. Next, we want to show that $m_X^{ij}Bdr_{\mathscr{A}}(A) \cap (X\backslash A) = \varnothing$. By Definition 3.1, we have

$$\begin{aligned} m_X^{ij}Bdr_{\mathscr{A}}(A) \cap (X\backslash A) &= (m_X^{ij}Cl_{\mathscr{A}}(A) \cap m_X^{ij}Cl_{\mathscr{A}}(X\backslash A)) \cap (X\backslash A) \\ &= A \cap (X\backslash A) \\ &= \varnothing. \end{aligned}$$

Therefore, $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq A$.

($\Leftarrow$) Let $m_X^{ij}Bdr_{\mathscr{A}}(A)$ be a subset of A. Then, $m_X^{ij}Bdr_{\mathscr{A}}(A) \cap (X\backslash A) = \varnothing$. Since $(X\backslash A) \subseteq m_X^{ij}Cl_{\mathscr{A}}(X\backslash A)$, $m_X^{ij}Cl_{\mathscr{A}}(A) \cap (X\backslash A) = \varnothing$, and finally, $m_X^{ij}Cl_{\mathscr{A}}(A) \subseteq A$. On the other hand, $A \subseteq m_X^{ij}Cl_{\mathscr{A}}(A)$. It follows that $m_X^{ij}Cl_{\mathscr{A}}(A) = A$. Moreover, A is (i, j)-m_X-α-closed.

(2) ($\Rightarrow$) Assume that A is (i, j)-m_X-α-open. Then, $m_X^{ij}Int_{\mathscr{A}}(A) = A$. Let us consider the following:

$$\begin{aligned} m_X^{ij}Bdr_{\mathscr{A}}(A) \cap A &= (m_X^{ij}Cl_{\mathscr{A}}(A) \cap m_X^{ij}Cl_{\mathscr{A}}(X\backslash A)) \cap A \\ &= A \cap m_X^{ij}Cl_{\mathscr{A}}(X\backslash A) \\ &= A \cap (X\backslash m_X^{ij}Int_{\mathscr{A}}(A)) \\ &= A \cap (X\backslash A) \\ &= \varnothing. \end{aligned}$$

Therefore, $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq X\backslash A$.

($\Leftarrow$) Assume that $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq X\backslash A$. It means that $m_X^{ij}Bdr_{\mathscr{A}}(A) \cap A = \varnothing$. Since $A \subseteq m_X^{ij}Cl_{\mathscr{A}}(A)$, $X\backslash m_X^{ij}Int_{\mathscr{A}}(A) \cap A = \varnothing$, and then $A \subseteq m_X^{ij}Int_{\mathscr{A}}(A)$. On the contrary, $m_X^{ij}Int_{\mathscr{A}}(A) \subseteq A$. Consequently, $m_X^{ij}Int_{\mathscr{A}}(A) = A$. Lastly, A is (i, j)-m_X-α-*open*. □

From Example 3.5, we can see that $m_X^{ij}Bdr_{\mathscr{A}}(A)$ are not necessary to be

$\varnothing$. For instance, $m_X^{12}Bdr_{\mathscr{A}}(\{2, 3\}) = \varnothing$, but $m_X^{21}Bdr_{\mathscr{A}}(\{2, 3\}) = \{1, 3\}$. All conditions to approach our purpose are found in the next theorem.

Theorem 3.7. *Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X. Then, $m_X^{ij}Bdr_{\mathscr{A}}(A) = \varnothing$ if and only if A is (i, j)-m_X-α-closed and (i, j)-m_X-α-open where $i, j = 1, 2$ and $i \neq j$.*

Proof. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X.

($\Rightarrow$) Assume that $m_X^{ij}Bdr_{\mathscr{A}}(A) = \varnothing$. Thus, $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq A$ and $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq X \backslash A$. By Theorem 3.6, A is (i, j)-m_X-*α-closed* and (i, j)-m_X-*α-open*.

($\Leftarrow$) Assume that A is (i, j)-m_X-*α-closed* and (i, j)-m_X-*α-open*. By Theorem 3.6, we also have $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq A$ and $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq (X \backslash A)$. As a result, $m_X^{ij}Bdr_{\mathscr{A}}(A) \subseteq A \cap (X \backslash A)$, and also $m_X^{ij}Bdr_{\mathscr{A}}(A) = \varnothing$. □

For the next section, we will introduce the concepts of (i, j)-m_X-α-exterior sets in biminimal structure space which contain some characterizations and several fundamental properties of those sets.

4. (i, j)-m_X-α-exterior Sets

Definition 4.1. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X and $x \in X$. Then, x is called (i, j)-m_X-α*-exterior point* of A if $x \in m_X^{ij}Int_{\mathscr{A}}(X \backslash A)$. We denote that the set of all (i, j)-m_X-α-exterior point of A by $m_X^{ij}Ext_{\mathscr{A}}(A)$, where $i, j = 1, 2$ and $i \neq j$.

According to Definition 4.1, $m_X^{ij}Ext_{\mathscr{A}}(A)$ can be rewritten as $X \backslash m_X^{ij}Cl_{\mathscr{A}}(A)$.

Example 4.2. Let $X = \{1, 2, 3\}$. Define m-structures m_X^1 and m_X^2 on the biminimal structure space X as follows: $m_X^1 = \{\varnothing, \{1, 3\}, \{2, 3\}, X\}$ and $m_X^2 = \{\varnothing, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, X\}$.

We have that $m_X^{12}Ext_{\mathscr{A}}(\{3\}) = X \backslash m_X^{12}Cl_{\mathscr{A}}(\{3\}) = X \backslash X = \varnothing$ and

$$m_X^{21}Ext_{\mathscr{A}}(\{3\}) = X \backslash m_X^{21}Cl_{\mathscr{A}}(\{3\}) = X \backslash \{3\} = \{1, 2\}.$$

Lemma 4.3. *Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X. Then, for any $i, j = 1, 2$ and $i \neq j$, the following statements hold:*

(1) $m_X^{ij}Ext_{\mathscr{A}}(A) \cap A = \varnothing$.

(2) $m_X^{ij}Ext_{\mathscr{A}}(\varnothing) = X$.

(3) $m_X^{ij}Ext_{\mathscr{A}}(X) = \varnothing$.

(4) $m_X^{ij}Ext_{\mathscr{A}}(X \backslash m_X^{ij}Ext_{\mathscr{A}}(A)) = m_X^{ij}Ext_{\mathscr{A}}(A)$.

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A is a subset of X.

(1) For the reason that $A \subseteq m_X^{ij}Cl_{\mathscr{A}}(A)$, then $X \backslash m_X^{ij}Cl_{\mathscr{A}}(A) \subseteq X \backslash A$. Furthermore, $(X \backslash m_X^{ij}Cl_{\mathscr{A}}(A)) \cap A \subseteq \varnothing$. That is $m_X^{ij}Ext_{\mathscr{A}}(A) \cap A = \varnothing$.

(2) Since $m_X^{ij}Cl_{\mathscr{A}}(\varnothing) = \varnothing$ and by Definition 4.1, we obtain that $m_X^{ij}Ext_{\mathscr{A}}(\varnothing) = X \backslash \varnothing = X$.

(3) Similar to (2) and the fact that $m_X^{ij}Cl_{\mathscr{A}}(X) = X$, we then have $m_X^{ij}Ext_{\mathscr{A}}(X) = X \backslash X = \varnothing$.

(4) By Definition 4.1,

$$m_X^{ij}Ext_{\mathscr{A}}(X\backslash m_X^{ij}Ext_{\mathscr{A}}(A)) = m_X^{ij}Ext_{\mathscr{A}}(m_X^{ij}Cl_{\mathscr{A}}(A))$$

and

$$m_X^{ij}Ext_{\mathscr{A}}(A) = X\backslash m_X^{ij}Cl_{\mathscr{A}}(m_X^{ij}Cl_{\mathscr{A}}(A)) = m_X^{ij}Ext_{\mathscr{A}}(m_X^{ij}Cl_{\mathscr{A}}(A)).$$

Then, $m_X^{ij}Ext_{\mathscr{A}}(X\backslash m_X^{ij}Ext_{\mathscr{A}}(A)) = m_X^{ij}Ext_{\mathscr{A}}(A)$. □

Theorem 4.4. *Let (X, m_X^1, m_X^2) be a biminimal structure space and A, B be subsets of X with $A \subseteq B$. Then, $m_X^{ij}Ext_{\mathscr{A}}(B) \subseteq m_X^{ij}Ext_{\mathscr{A}}(A)$, where $i, j = 1, 2$ and $i \neq j$.*

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A, B are subsets of X, $A \subseteq B$. Since $m_X^{ij}Cl_{\mathscr{A}}(A) \subseteq m_X^{ij}Cl_{\mathscr{A}}(B)$, we now have $X\backslash m_X^{ij}Cl_{\mathscr{A}}(B) \subseteq X\backslash m_X^{ij}Cl_{\mathscr{A}}(A)$. It follows that $m_X^{ij}Ext_{\mathscr{A}}(B) \subseteq m_X^{ij}Ext_{\mathscr{A}}(A)$ for any $i, j = 1, 2$ and $i \neq j$. □

Example 4.5. Let $X = \{1, 2, 3\}$. Define m-structures m_X^1 and m_X^2 on the biminimal structure space X as follows: $m_X^1 = \{\varnothing, \{1\}, \{2, 3\}, X\}$ and $m_X^2 = \{\varnothing, \{2\}, \{1, 3\}, X\}$.

We can see that $m_X^{12}Ext_{\mathscr{A}}(\{1, 2\}) \subseteq m_X^{12}Ext_{\mathscr{A}}(\{1\})$ and $m_X^{21}Ext_{\mathscr{A}}(\{1, 2\}) \subseteq m_X^{21}Ext_{\mathscr{A}}(\{1\})$, which $\{1\}$ is a subset of $\{1, 2\}$. Moreover, we also obtain that $m_X^{12}Ext_{\mathscr{A}}(\{3\}) = \{1\} \neq X\backslash\{3\}$, $m_X^{21}Ext_{\mathscr{A}}(\{3\}) = \{2\} \neq X\backslash\{3\}$, $m_X^{12}Ext_{\mathscr{A}}(X\backslash\{3\}) = \varnothing \neq \{3\}$ and $m_X^{21}Ext_{\mathscr{A}}(X\backslash\{3\}) = \varnothing \neq \{3\}$. Therefore, we need some conditions to complete those statements.

Theorem 4.6. *Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X. Then, for any $i, j = 1, 2$ and $i \neq j$, the following statements are true:*

(1) *A is (i, j)-m_X-α-closed if and only if $m_X^{ij}Ext_{\mathscr{A}}(A) = X\backslash A$.*

(2) *A is (i, j)-m_X-α-open if and only if $m_X^{ij}Ext_{\mathscr{A}}(X\backslash A) = A$.*

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A is a subset of X.

(1) ($\Rightarrow$) Suppose that A is (i, j)-m_X-α-closed. Then, $m_X^{ij}Ext_{\mathscr{A}}(A) = X\backslash m_X^{ij}Cl_{\mathscr{A}}(A) = X\backslash A$.

($\Leftarrow$) Suppose that $m_X^{ij}Ext_{\mathscr{A}}(A) = X\backslash A$. It means that $X\backslash m_X^{ij}Cl_{\mathscr{A}}(A) = X\backslash A$. Since $A \subseteq m_X^{ij}Cl_{\mathscr{A}}(A)$, $m_X^{ij}Cl_{\mathscr{A}}(A) = A$. Finally, A is (i, j)-m_X-α-closed.

(2) ($\Rightarrow$) Suppose that A is (i, j)-m_X-α-open. Then, $X\backslash A$ is (i, j)-m_X-α-closed. Using (1), $m_X^{ij}Ext_{\mathscr{A}}(X\backslash A) = X\backslash(X\backslash A) = A$.

($\Leftarrow$) Suppose that $m_X^{ij}Ext_{\mathscr{A}}(X\backslash A) = A$. We have

$$A = X\backslash m_X^{ij}Cl_{\mathscr{A}}(X\backslash A) = X\backslash(X\backslash m_X^{ij}Int_{\mathscr{A}}(A)) = m_X^{ij}Int_{\mathscr{A}}(A).$$

Hence, A is (i, j)-m_X-α-open. □

From Example 3.5, we have $m_X^{12}Ext_{\mathscr{A}}(\{1, 2\}) \cup m_X^{12}Ext_{\mathscr{A}}(\{3\}) = \{1\} \neq m_X^{12}Ext_{\mathscr{A}}(\{1, 2\} \cap \{3\})$, whereas $m_X^{12}Ext_{\mathscr{A}}(\{1\}) \cup m_X^{12}Ext_{\mathscr{A}}(\{2, 3\}) = X = m_X^{12}Ext_{\mathscr{A}}(\{1\} \cap \{2, 3\})$. So,

$$m_X^{ij}Ext_{\mathscr{A}}(A) \cup m_X^{ij}Ext_{\mathscr{A}}(B) \text{ and } m_X^{ij}Ext_{\mathscr{A}}(A \cap B)$$

are not necessary to be equal. In next theorem, we will give some condition for those sets to be equal.

Theorem 4.7. *Let* (X, m_X^1, m_X^2) *be a biminimal structure space, A, B be subsets of X. Then, for any* $i, j = 1, 2$ *and* $i \neq j$, *we have*

(1) $m_X^{ij}Ext_{\mathscr{A}}(A \cup B) \subseteq m_X^{ij}Ext_{\mathscr{A}}(A) \cap m_X^{ij}Ext_{\mathscr{A}}(B)$,

(2) $m_X^{ij}Ext_{\mathscr{A}}(A) \cup m_X^{ij}Ext_{\mathscr{A}}(B) \subseteq m_X^{ij}Ext_{\mathscr{A}}(A \cap B)$,

(3) *if A and B are* (i, j)*-*m_X*-*α*-closed, then*

$$m_X^{ij}Ext_{\mathscr{A}}(A) \cup m_X^{ij}Ext_{\mathscr{A}}(B) = m_X^{ij}Ext_{\mathscr{A}}(A \cap B).$$

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space, A, B are subsets of X.

(1) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by Theorem 4.4, $m_X^{ij}Ext_{\mathscr{A}}(A \cup B) \subseteq m_X^{ij}Ext_{\mathscr{A}}(A)$ and $m_X^{ij}Ext_{\mathscr{A}}(A \cup B) \subseteq m_X^{ij}Ext_{\mathscr{A}}(B)$. Therefore, $m_X^{ij}Ext_{\mathscr{A}}(A \cup B) \subseteq m_X^{ij}Ext_{\mathscr{A}}(A) \cap m_X^{ij}Ext_{\mathscr{A}}(B)$.

(2) By using Theorem 4.4 and the fact that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we then have $m_X^{ij}Ext_{\mathscr{A}}(A) \subseteq m_X^{ij}Ext_{\mathscr{A}}(A \cap B)$ and $m_X^{ij}Ext_{\mathscr{A}}(B) \subseteq m_X^{ij}Ext_{\mathscr{A}}(A \cap B)$. Lastly, $m_X^{ij}Ext_{\mathscr{A}}(A) \cup m_X^{ij}Ext_{\mathscr{A}}(B) \subseteq m_X^{ij}Ext_{\mathscr{A}}(A \cap B)$.

(3) Assume that A and B are (i, j)-m_X-α-closed. Therefore, $A \cap B$ is also (i, j)-m_X-α-closed. By Theorem 4.6, $m_X^{ij}Ext_{\mathscr{A}}(A \cap B) = X \setminus (A \cap B)$ $= (X \setminus A) \cup (X \setminus B) = m_X^{ij}Ext_{\mathscr{A}}(A) \cup m_X^{ij}Ext_{\mathscr{A}}(B)$. □

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